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Construction of Real-Valued Wavelets by Symmetry*

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A construction of orthogonal wavelet bases in $L_2(R^d)$ from a multiresolution analysis is given when the scaling function is skew-symmetric about an integer point. These wavelets are real-valued when the scaling function is real-valued. As an application, orthogonal wavelets generated by polyharmonic B-splines are obtained. We also present an example to show that the center of a skew-symmetric scaling function may not be in $Z^d/2$. © 1995 Academic Press, Inc.

The purpose of this paper is to give a construction of wavelet decomposition in the multivariate situation. Our construction relies on the multiresolution analyses whose scaling functions are skew-symmetric.

We say a sequence of closed subspaces $\{V_j\}_{j\in Z}$ of $L_2(R^d)$ forms a multiresolution analysis if it satisfies the following conditions:

(i) $V_j \subset V_{j+1}$, $j \in Z$;

(ii) $f \in V_j$ if and only if $f(2\,\cdot) \in V_{j+1}$;

(iii) $\overline{\bigcup_{j\in Z} V_j} = L_2(R^d)$;

(iv) $\bigcap_{j\in Z} V_j = \{0\}$;

(v) there exists a function $\phi \in V_0$ called a scaling function such that $\{\phi(\cdot - n)\}_{n\in Z^d}$ is an orthonormal basis of V_0.

A basic and important problem in the study of wavelet decomposition is the construction of orthogonal wavelet bases from a multiresolution analysis. The wavelet space W is defined as the orthogonal complement of V_0 in V_1. Our goal is to find a subset Ψ of W such that $\{\psi(\cdot - \alpha):\psi \in \Psi,\ \alpha \in Z^d\}$ is an orthonormal basis of W. Then $\{2^{kd/2}\psi(2^k \cdot - \alpha): \psi \in \Psi,\ k \in Z,\ \alpha \in Z^d\}$ forms an orthonormal basis for $L_2(R^d)$ called a wavelet basis.

There has been an extensive study of the construction of orthogonal wavelet bases in the multivariate case $d > 1$. A general construction under

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the weakest assumption was given by Jia and Shen in [5] recently. The disadvantage of their construction is that we can say nothing about the decay of the resulting wavelets. To have some control of the decay of the wavelets the intrinsic properties of the scaling function ϕ of the multiresolution analysis must be required. In this paper we are interested in skew-symmetric scaling functions satisfying

$$\phi(c_\phi + \cdot\,) = \bar{\phi}(c_\phi - \cdot\,) \qquad \text{for} \quad c_\phi \in R^d. \tag{1}$$

The first construction of orthogonal wavelets for such scaling functions was given in the low dimensions ($d \leqslant 3$) by Riemenschneider and Shen in [8, 9]. In arbitrary dimension $d \in N$ Jia and Shen gave a simple construction in [5]. However, as was stated by Jia and Shen, "If ϕ is symmetric about the origin, the wavelets constructed are complex-valued in general." The purpose of this paper is to overcome this shortcoming and construct orthogonal wavelets which are real-valued if ϕ is real-valued and skew-symmetric about some point $c_\phi \in Z^d$.

Suppose that $\phi \in L_2(R^d)$ is the scaling function of a multiresolution analysis satisfying (1) for some point $c_\phi \in Z^d$. Without loss of generality, we may assume that $c_\phi = 0$. Since ϕ is refinable, we have

$$\phi(\,\cdot\,) = \sum_{\alpha \in Z^d} b(\alpha)\, \phi(2\cdot - \alpha) \tag{2}$$

for some sequence $\{b(\alpha)\} \in l_2(Z^d)$.

Note that ϕ has orthonormal integer translates; we have

$$b(\alpha) = \bar{b}(-\alpha) \tag{3}$$

for all $\alpha \in Z^d$.

Throughout this paper we shall assume that $\{b(\alpha)\} \in l_1(Z^d)$ and $\sum_{\alpha \in Z^d} b(\alpha) = 2^d$. This assumption is valid if ϕ satisfies $\sum_{\alpha \in Z^d} |\phi(\cdot - \alpha)| \in L_2([0, 1)^d)$, see [11].

Let $\mathscr{E} = \mathscr{E}_d$ be the set of all extreme points of the unit cube $[0, 1]^d$, i.e.,

$$\mathscr{E} = \mathscr{E}_d := \{(v_1, ..., v_d) : v_j = 0 \text{ or } 1 \text{ for all } j\}.$$

For $\mu \in \mathscr{E}$, we define

$$\phi_\mu = 2^{d/2}\phi(2\cdot - \mu)$$

and

$$b_\mu(\beta) = 2^{-d/2} b(2\beta + \mu).$$

Then we know that $\{\phi_\mu(\cdot - \alpha) : \mu \in \mathscr{E},\ \alpha \in Z^d\}$ is an orthonormal basis of V_1. From (2) we also have

$$\hat{\phi}(\omega) = \sum_{\mu \in \mathscr{E}} \tilde{b}_\mu(e^{-i\omega})\, \hat{\phi}_\mu(\omega), \qquad \omega \in R^d. \tag{4}$$

Here

$$\hat{\phi}(\omega) = \int_{R^d} \phi(x)\, e^{-ix \cdot \omega}\, dx$$

is the Fourier transform of ϕ, $x \cdot \omega$ denotes the inner product of the two vectors x and ω in R^d, and $\tilde{b}_\mu$ is the Laurent series called the symbol of b_μ given by

$$\tilde{b}_\mu(z) = \sum_{\alpha \in Z^d} b_\mu(\alpha)\, z^\alpha.$$

Suppose we are given a wavelet set $\{\psi_\mu : \mu \in \mathscr{E} \backslash \{0\}\}$. Each ψ_μ has a representation of the form

$$\psi_\mu = \sum_{\nu \in \mathscr{E}} \sum_{\alpha \in Z^d} b_{\mu\nu}(\alpha)\, \phi_\nu(\cdot - \alpha), \tag{5}$$

where $b_{\mu\nu} \in l_2(Z^d)$ $(\mu \in \mathscr{E} \backslash \{0\},\ \nu \in \mathscr{E})$.

Let $b_{0,\nu} = b_\nu$, $\nu \in \mathscr{E}$. Then we have the following characterization of a wavelet set.

LEMMA 1. *The set $\{\psi_\mu(\cdot - \alpha) : \mu \in \mathscr{E} \backslash \{0\},\ \alpha \in Z^d\}$ forms an orthonormal basis of W if and only if $(\tilde{b}_{\mu\nu}(z))_{\mu,\nu \in \mathscr{E}}$ is a unitary matrix for almost every $z \in T^d = \{(z_1, ..., z_d) \in C^d : |z_1| = \cdots = |z_d| = 1\}$.*

The proof of this lemma can be found in [3, 6].

Thus the problem of wavelet decomposition is reduced into the problem of completing a unitary matrix with the first row given. Any wavelet decomposition depends on some intrinsic properties of the first row. In the case that the scaling function is skew-symmetric we have the following property.

LEMMA 2. *Suppose that $\phi \in L_2(R^d)$ is a scaling function and skew-symmetric about the origin point. Let the mask sequence $\{b(\alpha)\}_{\alpha \in Z^d}$ of the refinement equation (2) satisfy $\{b(\alpha)\} \in l_1(Z^d)$ and $\sum_{\alpha \in Z^d} b(\alpha) = 2^d$. Then for any $z \in T^d$,*

$$(\tilde{b}_\mu(z))_{\mu \in \mathscr{E}} \neq (-2^{-d/2}, ..., -2^{-d/2}).$$

Proof. Suppose to the contrary that there exists $z_0 \in T^d$ such that $\tilde{b}_\mu(z_0) = -2^{-d/2}$ for all $\mu \in \mathscr{E}$.

By (3) we have for $\mu \in \mathscr{E}$ and $z \in T^d$

$$\begin{aligned}\tilde{b}_\mu(z) &= 2^{-d/2} \sum_{\alpha \in Z^d} b(\mu + 2\alpha)\, z^\alpha \\ &= 2^{-d/2} \sum_{\alpha \in Z^d} \overline{b(\mu - 2\alpha - 2\mu)\, z^{-\alpha}} \\ &= z^{-\mu} \overline{\tilde{b}_\mu(z)}.\end{aligned}$$

Thus, for all $\mu \in \mathscr{E}$, we have

$$z_0^\mu = 1,$$

which implies that $z_0 = (1, ..., 1)$.

By our conditions and assumptions

$$\begin{aligned}\sum_{\mu \in \mathscr{E}} \tilde{b}_\mu(z_0) &= -2^{d/2} \\ &= \sum_{\mu \in \mathscr{E}} \sum_{\alpha \in Z^d} 2^{-d/2} b(\mu + 2\alpha) \\ &= 2^{-d/2} \sum_{\alpha \in Z^d} b(\alpha) \\ &= 2^{d/2}.\end{aligned}$$

This is a contradiction. Hence we cannot have $z \in T^d$ such that $\tilde{b}_\mu(z) = -2^{-d/2}$ for all $\mu \in \mathscr{E}$. The proof of Lemma 2 is complete. ∎

With the above property we can now give the extensibility of the mask symbol.

For $n \in N$, we let I be the $n \times n$ identity matrix, e the $n \times 1$ vector $(1, 0, ..., 0)^T$, where we denote A^T as the transpose of the matrix A, and A^* the complex conjugate transpose of A.

Given a nonzero $n \times 1$ vector v in C^n, we define the Householder matrix $H(v)$ as

$$H(v) = I - 2vv^*/(v^*v). \tag{6}$$

We also need a Householder-type matrix given in [4, 5] as

$$Q(z) = -I + (e + \bar{z})(e + z)^T/(1 + \bar{z}_1), \tag{7}$$

where $z = (z_1, ..., z_n)^T \in C^n$ and $z_1 \neq -1$.

LEMMA 3. *Let $u = (-1/\sqrt{n}, ..., -1/\sqrt{n})^T$. For $z \in C^n$ with $\sum_{j=1}^n |z_j|^2 = 1$ and $z \neq u$, we let*

$$P(z) = -Q(-H(u - e)z)\, H(u - e). \tag{8}$$

Then $P(z)$ is a unitary matrix with z^T as its first row.

Proof. It is easily checked that $H(v)$ is unitary and Hermitian for any nonzero vector v. From [5, Lemma 3.3] we also have

$$-H(u-e)u = -e.$$

We need a result of Jia and Shen [5, Theorem 5.1]:

$Q(z)$ is a unitary matrix with $(z_1, ..., z_n) = z^T$ as its first row whenever $\sum_{j=1}^{n} |z_j|^2 = 1$ and $z \neq -e$.

Note that $\sum_{j=1}^{n} |z_j|^2 = 1 = z^* z$. We obtain

$$(-H(u-e)z)^* (-H(u-e)z) = z^*(H(u-e))^* H(u-e)z$$
$$= z^* z = 1.$$

By our assumption, $z \neq u$; hence

$$-H(u-e)z \neq -e.$$

Therefore, $Q(-H(u-e)z)$ is a unitary matrix and its first row is

$$(-H(u-e)z)^T = -z^T(H(u-e))^T.$$

Thus, $P(z)$ is a unitary matrix and its first row is

$$-(-z^T(H(u-e))^T H(u-e)) = z^T.$$

The proof of Lemma 3 is complete. ∎

Now we can give our construction of a wavelet set.

THEOREM 1. *Suppose that $\phi \in L_2(R^d)$ is a scaling function and skew-symmetric about the origin point. Let the mask sequence $\{b(\alpha)\}_{\alpha \in Z^d}$ of the refinement equation* (2) *satisfy $\{b(\alpha)\} \in l_1$ and $\sum_{\alpha \in Z^d} b(\alpha) = 2^d$. Then choosing for $z \in T^d$*

$$\tilde{b}_{\mu\nu}(z) = \begin{cases} -2^{-d/2} + \left\{2^{-d/2}\overline{\tilde{b}_0(z)} - \overline{\tilde{b}_\mu(z)} + \dfrac{1}{2^{d/2}+2^d} \displaystyle\sum_{\eta \in \mathscr{E}\setminus\{0\}} \overline{\tilde{b}_\eta(z)}\right\} \\ \quad \times \{2^{-d/2} + \tilde{b}_0(z)\} \Big/ \left\{1 + 2^{-d/2} \displaystyle\sum_{\eta \in \mathscr{E}} \overline{\tilde{b}_\eta(z)}\right\}, \\ \quad \mu \in \mathscr{E}\setminus\{0\},\ \nu = 0; \\ \delta_{\mu\nu} - \dfrac{1}{2^{d/2}+2^d} + \left\{2^{-d/2}\overline{\tilde{b}_0(z)} - \overline{\tilde{b}_\mu(z)} + \dfrac{1}{2^{d/2}+2^d} \displaystyle\sum_{\eta \in \mathscr{E}\setminus\{0\}} \overline{\tilde{b}_\eta(z)}\right\} \\ \quad \times \{2^{-d/2} + \tilde{b}_\nu(z)\} \Big/ \left\{1 + 2^{-d/2} \displaystyle\sum_{\eta \in \mathscr{E}} \overline{\tilde{b}_\eta(z)}\right\}, \\ \quad \mu, \nu \in \mathscr{E}\setminus\{0\}, \end{cases} \tag{9}$$

we have a wavelet set $\{\psi_\mu : \mu \in \mathscr{E}\backslash\{0\}\}$ *such that* $\{\psi_\mu(\cdot - \alpha) : \mu \in \mathscr{E}\backslash\{0\}, \alpha \in Z^d\}$ *is an orthonormal basis of* W *if we define* ψ_μ *as in* (5).

The proof of Theorem 1 can be easily given from the above lemmas.

We mention only that when ϕ is real-valued, $\{b(\alpha)\}$ is real-valued and for $z \in T^d$ and $\mu \in \mathscr{E}$

$$\overline{\tilde{b}_\mu(z)} = \sum_{\alpha \in Z^d} 2^{-d/2} \overline{b(\mu + 2\alpha)}\, z^{-\alpha}$$

$$= \sum_{\alpha \in Z^d} 2^{-d/2} b(\mu - 2\alpha)\, z^{\alpha}.$$

Hence the coefficient sequence $b_{\mu\nu}$ in the Laurent series of $\tilde{b}_{\mu\nu}(z)(\mu, \nu \in \mathscr{E})$ is real-valued and therefore the wavelet set $\{\psi_\mu\}$ defined by (5) is real-valued.

We note that the wavelets constructed in Theorem 1 decay exponentially fast if the scaling function ϕ does so. By Wiener's lemma we can also see that the coefficient sequence $b_{\mu\nu}$ given by (9) is in $l_1(Z^d)$.

For box-splines, Theorem 1 gives another construction of orthogonal wavelet sets different from those of Jia and Shen in [5] and Stöckler in [10].

As another application we consider the polyharmonic B-splines.

The polyharmonic B-spline N in R^d, normalized to have integral one, has a Fourier transform given by

$$\hat{N}(\omega) = 2^{2r} \left(\sum_{j=1}^{d} \sin^2 \omega_j/2 \right)^r \Bigg/ \left(\sum_{j=1}^{d} \omega_j^2 \right)^r, \qquad \omega \in R^d, \tag{10}$$

where we assume $r > d/2$, see [7].

Define $\phi \in L_2(R^d)$ as

$$\hat{\phi}(\omega) = \hat{N}(\omega) \Bigg/ \left(\sum_{\alpha \in Z^d} |\hat{N}(\omega + 2\pi\alpha)|^2 \right)^{1/2}, \qquad \omega \in R^d. \tag{11}$$

Then ϕ is the scaling function of a multiresolution analysis. For $x \in R^d$ we have $\phi(x) = \phi(-x) = \overline{\phi(x)}$. That is, ϕ is real-valued and skew-symmetric about the origin point and we can apply Theorem 1 to obtain an orthogonal wavelet set.

THEOREM 2. *Let* N *and* ϕ *be defined as in* (10) *and* (11); *then* ϕ *is the scaling function of a multiresolution analysis* $\{V_j\}_{j \in Z}$ *of* $L_2(R^d)$. *Define* $\{\psi_\mu : \mu \in \mathscr{E}\backslash\{0\}\}$ *as in Theorem* 1; *then* $\{\psi_\mu(\cdot - \alpha) : \mu \in \mathscr{E}\backslash\{0\}, \alpha \in Z^d\}$ *is an orthonormal real-valued wavelet basis of* W, *the orthogonal complement of* V_0 *in* V_1.

Let us mention that other constructions are possible if we replace the vector in Theorem 1 $(-2^{-d/2}, ..., -2^{-d/2})$ by other real-valued vectors. The method of construction is similar. The construction for $d=1$ can be found in [1, 2].

If a scaling function ϕ of a multiresolution analysis in $L_2(R^d)$ satisfies (1), then we may hope that the center c_ϕ of ϕ will be in $Z^d/2$, since in this case we can construct the orthogonal wavelet bases by [5] and our Theorem 1. However, in the second part of this paper we will present an example to show that this is not true.

Let $d=1$. We consider Meyer's well-known basis: $\phi_1 \in L_2(R)$ satisfies $\hat{\phi}_1(\omega)=1$ for $|\omega| \leqslant \frac{2}{3}\pi$; $\hat{\phi}_1(\omega)=0$ for $|\omega| \geqslant \frac{4}{3}\pi$; $0 \leqslant \hat{\phi}_1(\omega) \leqslant 1$ for $\frac{2}{3}\pi \leqslant |\omega| \leqslant \frac{4}{3}\pi$; $\hat{\phi}_1 \in C^\infty(R)$ is even and satisfies $|\hat{\phi}_1(\omega)|^2 + |\hat{\phi}_1(2\pi-\omega)|^2 = 1$ for $\frac{2}{3}\pi \leqslant \omega \leqslant \frac{4}{3}\pi$.

We know that

$$\hat{\phi}_1(\omega) = H_1(\omega/2)\,\hat{\phi}_1(\omega/2), \qquad \omega \in R, \tag{12}$$

satisfies for a 2π-periodic C^∞ function H_1.

Now we define $\phi \in L_2(R)$ as

$$\phi(x) = \phi_1(x-a), \tag{13}$$

where $a \in R$ is arbitrarily chosen.

Then $\phi \in \mathscr{L}_2(R) \subset L_1(R)$, i.e., $\sum_{k \in Z} |\phi(\cdot + k)| \in L_2([0,1))$. ϕ is real-valued since ϕ_1 is. The integer translates of ϕ are orthonormal since $\hat{\phi}(\omega) = e^{-ia\omega}\widehat{\phi_1}(\omega)$ and

$$\sum_{k \in Z} |\hat{\phi}(\omega + 2k\pi)|^2 = \sum_{k \in Z} |\widehat{\phi_1}(\omega + 2k\pi)|^2 = 1$$

for any $\omega \in R$.

Let $H(\omega)$ be the 2π-periodic function defined by

$$H(\omega) = e^{-ia\omega} H_1(\omega) \tag{14}$$

for $|\omega| \leqslant \pi$.

THEOREM 3. *Let ϕ and H be defined as above. Then ϕ is the scaling function of a multiresolution analysis, skew-symmetric about $a \in R$ and the mask sequence of the refinement equation which ϕ satisfies is in $l_1(Z)$.*

Proof. We show that for any $\omega \in R$,

$$\hat{\phi}(\omega) = H\left(\frac{\omega}{2}\right) \hat{\phi}\left(\frac{\omega}{2}\right). \tag{15}$$

If $|\omega| \geqslant 4\pi$, then

$$\hat{\phi}(\omega) = \hat{\phi}\left(\frac{\omega}{2}\right) = 0$$

$$= \widehat{\phi_1}(\omega) = \widehat{\phi_1}\left(\frac{\omega}{2}\right).$$

If $|\omega| \leqslant 2\pi$, then by (12) we have

$$\hat{\phi}(\omega) = e^{-ia\omega}\widehat{\phi_1}(\omega)$$

$$= e^{-ia\omega} H_1\left(\frac{\omega}{2}\right) \widehat{\phi_1}\left(\frac{\omega}{2}\right)$$

$$= H\left(\frac{\omega}{2}\right) \hat{\phi}\left(\frac{\omega}{2}\right)$$

since $|\omega/2| \leqslant \pi$.

If $2\pi \leqslant |\omega| \leqslant 4\pi$ and $\widehat{\phi_1}(\omega/2) = 0$, then

$$\hat{\phi}(\omega) = e^{-ia\omega}\widehat{\phi_1}(\omega)$$

$$= 0 = H\left(\frac{\omega}{2}\right) \hat{\phi}\left(\frac{\omega}{2}\right)$$

since

$$\hat{\phi}\left(\frac{\omega}{2}\right) = e^{-ia(\omega/2)}\widehat{\phi_1}\left(\frac{\omega}{2}\right) = 0.$$

The left case is $2\pi \leqslant |\omega| \leqslant 4\pi$ and $\widehat{\phi_1}(\omega/2) \neq 0$. In this case, by (12) we have $H_1(\omega/2) = 0 = H_1(\omega/2 \pm 2\pi)$. If $2\pi \leqslant \omega \leqslant 4\pi$, then $\omega/2 - 2\pi \in [-\pi, \pi]$ and

$$H\left(\frac{\omega}{2}\right) = H\left(\frac{\omega}{2} - 2\pi\right)$$

$$= e^{-ia(\omega/2 - 2\pi)} H_1\left(\frac{\omega}{2} - 2\pi\right) = 0.$$

Hence (15) is also valid.

The proof for $-4\pi \leqslant \omega \leqslant -2\pi$ is similar and we have proved (15) for any $\omega \in R$. We note that $H(\omega)$ is continuous in fact.

Thus, from (15) we know that ϕ is refinable and has orthonormal integer translates. Also, $\sum_{k \in Z} |\phi(\cdot + k)| \in L_2([0, 1))$. By [11, Theorem 2] we know that

$$H(\omega) = \tfrac{1}{2}\tilde{b}(e^{-i\omega}), \qquad \omega \in R,$$

for some sequence $b \in l_1(Z)$.

Then by a result of Jia and Micchelli [3], we see that ϕ is the scaling function of a multiresolution analysis of $L_2(R)$. It is evident that ϕ is real-valued and skew-symmetric about $a \in R$.

The proof of Theorem 3 is complete. ∎

If $\hat{\phi}(\omega) \neq 0$ for a.e. $\omega \in R^d$, then it is true that for a skew-symmetric scaling function ϕ, the center must be in $Z^d/2$.

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